

Almost sure-sign convergence of Hardy-type Dirichlet series

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Abstract

Hartman proved in 1939 that the width of the largest possible strip in the complex plane, on which a Dirichlet series $\sum_n a_n n^{-s}$ is uniformly a.s.-sign convergent (i.e., $\sum_n \varepsilon_n a_n n^{-s}$ converges uniformly for almost all sequences of signs $\varepsilon_n = \pm 1$) but does not converge absolutely, equals $1/2$. We study this result from a more modern point of view within the framework of so-called Hardy-type Dirichlet series with values in a Banach space

1 Introduction

The natural domains of convergence of Dirichlet series are half-planes. Given a Dirichlet series $D = \sum a_n n^{-s}$ there are three abscissas which define the biggest half-planes on which D converges, converges uniformly and converges absolutely:

$$\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D). \quad (1)$$

Whereas it is not difficult to show that

$$\sup_{D \text{ Dir. ser.}} \sigma_a(D) - \sigma_u(D) = 1, \quad (2)$$

the main issue in the 1910's was to decide what the maximal width of the band on which a Dirichlet series can converge uniformly but not absolutely is. This problem was first considered by Harald Bohr and consists on computing the following number

$$S := \sup_{D \text{ Dir. ser.}} \sigma_a(D) - \sigma_u(D).$$

Bohr himself [8] showed in 1913 that $S \leq 1/2$, but it was not until 1931 that Bohnenblust and Hille [7] proved that indeed

$$S = 1/2. \quad (3)$$

The proof of the lower bound for S by Bohnenblust and Hille is long and involved. A few years later Hartman gave in [19] a different proof for the lower bound, based on probabilistic arguments. Let us be more precise. On $\{-1, 1\}$ we consider the probability $\mathbf{P}(-1) = \mathbf{P}(1) = 1/2$ and on $\{-1, 1\}^{\mathbb{N}}$ its product probability. From now on $(\varepsilon_n)_n$ will always be a sequence of signs in $\{-1, 1\}^{\mathbb{N}}$. We say that $D = \sum a_n n^{-s}$ is (uniformly) a.s.-sign convergent on a half plane $[\operatorname{Re} > \sigma]$ whenever $D = \sum \varepsilon_n a_n n^{-s}$ (uniformly) converges on $[\operatorname{Re} > \sigma]$ outside of a zero set of signs ε_n .

Given a Dirichlet series $D = \sum a_n n^{-s}$, Hartman in [19] (with a slightly different notation) considers the following abscissas

$$\begin{aligned} \sigma_c^{\text{rad}}(D) &:= \inf \left\{ \sigma \in \mathbb{R} : \sum a_n n^{-s} \text{ a.s.-sign convergent on } [\operatorname{Re} > \sigma] \right\} \\ \sigma_u^{\text{rad}}(D) &:= \inf \left\{ \sigma \in \mathbb{R} : \sum a_n n^{-s} \text{ uniformly a.s.-sign convergent on } [\operatorname{Re} > \sigma] \right\} \end{aligned}$$

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(obviously, it doesn't make any sense to define an analogous notion like $\sigma_a^{\text{rad}}(D)$). In general the abscissas $\sigma_c^{\text{rad}}(D)$ and $\sigma_u^{\text{rad}}(D)$ are different from $\sigma_c(D)$ and $\sigma_u(D)$, respectively. The two main result in Hartman's article are (compare with (2) and (3))

$$\sup_{D \text{ Dir. ser.}} \sigma_a(D) - \sigma_c^{\text{rad}}(D) = \frac{1}{2} \quad (4)$$

and

$$\sup_{D \text{ Dir. ser.}} \sigma_a(D) - \sigma_u^{\text{rad}}(D) = \frac{1}{2}. \quad (5)$$

Hartman in particular proves that $\sup_D \sigma_a(D) - \sigma_u^{\text{rad}}(D) \leq \sup_D \sigma_a(D) - \sigma_u(D)$, and estimating the first sup from below he produces a substantially different proof of the lower bound of S . His proof is probabilistic and goes through almost periodic functions with random Fourier coefficients.

Recently, many authors have shown new interest in the Bohr-Bohnenblust-Hille circle of ideas (see the recent monograph [26] and also [1, 2, 3, 5, 6, 4, 9, 11, 13, 12, 15, 17, 20, 22, 24, 25]), and probabilistic arguments have shown to be of great interest in this theory. Being Hartman's paper the first time when such probabilistic arguments were used to deal with Dirichlet series, our aim in this note is to look at his results from this more modern and more general point of view that we believe clarifies the original argument.

Vector-valued Hardy-type Dirichlet series We are going to work with Hardy spaces of Dirichlet series with values in a Banach space. In this way we continue and extend our work from [9]. Given a Banach space E , we consider the one-to-one correspondence between the spaces $\mathfrak{P}(E)$ (all formal power series $\sum_{\alpha} c_{\alpha} z^{\alpha}$ in infinitely many variables with coefficients $c_{\alpha} \in E$) and $\mathfrak{D}(E)$ (all formal Dirichlet series $\sum_n a_n \frac{1}{n^s}$ with coefficients $a_n \in E$)

$$\mathfrak{P}(E) \longrightarrow \mathfrak{D}(E), \quad \sum c_{\alpha} z^{\alpha} \mapsto \sum a_n n^{-s} \quad (6)$$

given by $a_n = c_{\alpha}$ if $n = p^{\alpha} = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_1 < p_2 < p_3 < \dots$ stands for the sequence of prime numbers (see [9]).

Let us recall the definition of Hardy spaces of E -valued Dirichlet series (first defined by Bayart for $E = \mathbb{C}$ in [3] and later for arbitrary E in [9]). For every $f \in L_1(\mathbb{T}^{\mathbb{N}}; E)$ (the Banach space of Bochner integrable E -valued functions defined on the infinite dimensional torus $\mathbb{T}^{\mathbb{N}}$ with the normalized Lebesgue measure dz) and every multi index $\alpha \in \mathbb{Z}^{(\mathbb{N})}$ (all finite sequences $\alpha = (\alpha_n)_{n \in \mathbb{N}}$) we as usual denote the α th Fourier coefficient of f by $\hat{f}(\alpha) = \int_{\mathbb{T}^{\mathbb{N}}} f(z) z^{-\alpha} dz$. Now define for $1 \leq p < \infty$ the Hardy space

$$H_p(\mathbb{T}^{\mathbb{N}}; E) := \{f \in L_p(\mathbb{T}^{\mathbb{N}}; E) : \hat{f}(\alpha) \neq 0 \text{ only if } \alpha \in \mathbb{N}_0^{(\mathbb{N})}\}$$

(with the norm induced by $L_p(\mathbb{T}^{\mathbb{N}}; E)$) and let

$$\mathcal{H}_p(E)$$

by definition be the image of the Banach space $H_p(\mathbb{T}^{\mathbb{N}}; E)$ by the aforementioned correspondence (6) (again with the norm coming from $H_p(\mathbb{T}^{\mathbb{N}}; E)$). We also consider

$$\mathcal{H}_{\infty}(E),$$

the space of E -valued Dirichlet series such that $\sum_n a_n \frac{1}{n^s}$ defines a bounded, holomorphic function on $[\text{Re } s > 0]$, with the norm $\|\sum a_n n^{-s}\|_{\mathcal{H}_{\infty}(E)} := \sup_{\text{Re } s > 0} \|\sum_n a_n \frac{1}{n^s}\|_E$. We note that this Banach space through the identification in (6) coincides isometrically with $H_{\infty}(\mathbb{T}^{\mathbb{N}}; E)$ if and only if E has the analytic Radon-Nikodym property (see [16]). In the scalar case we abbreviate

$$\mathcal{H}_p = \mathcal{H}_p(\mathbb{C}), \quad 1 \leq p \leq \infty.$$

Clearly, we have that

$$\mathcal{H}_2 = \left\{ \sum a_n n^{-s} : \|D\|_{\mathcal{H}_2} = \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} < \infty \right\}, \quad (7)$$

a Hilbert space intensively studied by Hedenmalm, Lindqvist and Seip in [20].

State of art. For any Banach space E and any $1 \leq p \leq \infty$ define the $\mathcal{H}_p(E)$ -abscissa of a Dirichlet series $\sum a_n n^{-s}$ by

$$\sigma_{\mathcal{H}_p(E)}(D) := \inf \left\{ \sigma \in \mathbb{R} : \sum \frac{a_n}{n^\sigma} n^{-s} \in \mathcal{H}_p(E) \right\}.$$

Then, given $1 \leq p \leq \infty$, we have (for $p = \infty$ see [7] (scalar case) and [13, Theorem 1] (vector-valued case), and for $1 \leq p < \infty$ see [2, 3] (scalar case) and [9] (vector-valued case))

$$S_p(E) := \sup_{D \in \mathfrak{D}(E)} \sigma_a(D) - \sigma_{\mathcal{H}_p(E)} = 1 - \frac{1}{\cot E}; \quad (8)$$

recall that E has cotype q (with $2 \leq q < \infty$) if there is a constant $C > 0$ such that for every finite choice of elements $x_1, \dots, x_N \in E$ we have $(\sum_k \|x_k\|_E^q)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{T}^N} \|\sum_k x_k z_k\|_E^2 dz \right)^{1/2}$, and

$$\cot E := \inf \left\{ q \in [2, \infty[: E \text{ has cotype } q \right\}.$$

Let us comment on the special case $E = \mathbb{C}$ for which $\cot \mathbb{C} = 2$. For $p = \infty$ we know by Bohr's fundamental theorem from [8] (see also [26, Theorem 6.2.3]) that

$$\sigma_u(D) = \sigma_{\mathcal{H}_\infty}(D), \quad (9)$$

hence (8) implies (3). For $p = 2$ we have

$$\sigma_c^{\text{rad}}(D) = \sigma_{\mathcal{H}_2}(D), \quad (10)$$

so in this case (8) implies (4). Indeed, by Khinchin's inequality it is well-known that a scalar sequence $x = (x_n)$ is a.s.-sign summable (i.e., $\sum_n \varepsilon_n x_n$ converges for almost all possible choices of signs ε_n) if and only if $x \in \ell_2$. This, together with (7), is what we need. We will come back to this issue later.

Is it also possible to recover (5) within the setting of Hardy-type Dirichlet series? Given $1 \leq p \leq \infty$ and a Banach space E , we define what is going to be one of our main objects,

$$\mathcal{H}_p^{\text{rad}}(E) := \left\{ \sum a_n n^{-s} : \forall \text{ a.e. } \varepsilon_n = \pm 1, \sum \varepsilon_n a_n n^{-s} \in \mathcal{H}_p(E) \right\}. \quad (11)$$

Then, for a given Dirichlet series $D \in \mathfrak{D}(E)$, we consider the abscissa

$$\sigma_{\mathcal{H}_p(E)}^{\text{rad}}(D) := \inf \left\{ \sigma \in \mathbb{R} : \sum \frac{a_n}{n^\sigma} n^{-s} \in \mathcal{H}_p^{\text{rad}}(E) \right\},$$

and again the aim is to determine the maximal distance between $\sigma_a(D)$ and $\sigma_{\mathcal{H}_p(E)}^{\text{rad}}(D)$.

Summary. Our first main result (Theorem 8) is a proper extension of Hartman's main result from (5) and an analogue of (8) in the setting of a.s.-sign convergence of Hardy-type Dirichlet series: For every Banach space E and $1 \leq p \leq \infty$

$$S_p^{\text{rad}}(E) := \sup_{D \in \mathfrak{D}(E)} \sigma_a(D) - \sigma_{\mathcal{H}_p(E)}^{\text{rad}}(D) = 1 - \frac{1}{\cot E}. \quad (12)$$

Indeed, (12) recovers (5) since $\cot \mathbb{C} = 2$ and $\sigma_{\mathcal{H}_\infty}^{\text{rad}}(D)$ is the abscissa $\sigma_u^{\text{rad}}(D)$ defined by Hartman. Moreover, we show that $S_p^{\text{rad}}(E) \leq S_p(E)$ (Corollary 7), hence (12) also recovers (8). For the proof of (12) we distinguish between finite and infinite dimensional Banach spaces E . In the rest of our article we graduate (12). Following an idea from [7], we give precise estimates for the m th graduation of $S_p^{\text{rad}}(E)$ along m -homogeneous Dirichlet series (see Proposition 9 and Proposition 10). In the scalar case, we graduate (12) along the length of the considered Dirichlet series; here our main results are Theorem 11 and Theorem 12. Finally, in the Appendix we show the remarkable fact that the maximum width of the strips of a.s.-sign but not absolute convergence and of uniform a.s.-sign but not absolute convergence coincide (see (4) and (5)) extends to the vector-valued case.

2 The Banach space $\mathcal{H}_p^{\text{rad}}(E)$

In this section we collect a few facts on $\mathcal{H}_p^{\text{rad}}(E)$ needed later. First of all, we need a norm on $\mathcal{H}_p^{\text{rad}}(E)$. We denote by $(r_n)_n$ the system of Rademacher functions on $[0, 1]$. We are going to use the following, fundamental for us, fact (see e.g. [18, Theorem 12.3]): Given a sequence $(x_n)_n$ in a Banach space X , the series $\sum_n r_n x_n$ converges almost surely if and only if $\sum_n r_n x_n$ converges in $L_p([0, 1]; X)$ for some (and then all) $0 < p < \infty$.

Clearly, $\sum_n r_n x_n$ converges a.e. is another way to say that $\sum_n \varepsilon_n x_n$ is a.s.-sign convergent. Then taking $X = \mathcal{H}_p(E)$ and $x_n = a_n n^{-s} \in \mathcal{H}_p(E)$ we can reformulate our space $\mathcal{H}_p^{\text{rad}}(E)$ defined in (11) as follows:

$$\mathcal{H}_p^{\text{rad}}(E) = \left\{ \sum a_n n^{-s} : \sum_n r_n a_n n^{-s} \in L_1([0, 1]; \mathcal{H}_p(E)) \right\},$$

and define the norm

$$\left\| \sum a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(E)} := \int_0^1 \left\| \sum_n r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_p(E)} dt. \quad (13)$$

We need $\mathcal{H}_p^{\text{rad}}(E)$ to be a Banach space. This follows from the following general result. First, recall (see [18, page 233]) that for a given Banach space X , the space $\text{Rad}(X)$ of almost unconditionally summable sequences $(x_n)_n$ in X together with the norm

$$\|(x_n)_n\|_{\text{Rad}(X)} = \int_0^1 \left\| \sum_n r_n(t) x_n \right\|_X dt$$

forms a Banach space.

Lemma 1. *Let X be a Banach space and let $Y_n, n \in \mathbb{N}$ be closed subspaces of X . Then $Y = \{(x_n)_n \in \text{Rad}(X) : x_n \in Y_n\}$ is closed in $\text{Rad}(X)$.*

Proof. Let us observe first that if $x = (x_n)_n \in \text{Rad}(X)$, then $\sum_m r_m x_m \in L_1([0, 1]; X)$. Due to the orthogonality of the Rademacher system we have

$$x_n = \sum_m x_m \int_0^1 r_m(t) r_n(t) dt = \int_0^1 \left(\sum_m x_m r_m(t) \right) r_n(t) dt,$$

and this gives

$$\|x_n\|_X \leq \int_0^1 \left\| \sum_m x_m r_m(t) \right\|_X |r_n(t)| dt = \int_0^1 \left\| \sum_m x_m r_m(t) \right\|_X dt = \|x\|_{\text{Rad}(X)}.$$

Let us take now $(x^{(m)})_m \in Y$ that converges to a certain x in $\text{Rad}(X)$. We write $x^{(m)} = (x_n^{(m)})_n$ and $x = (x_n)_n$, then $\|x_n^{(m)} - x_n\|_X \leq \|x^{(m)} - x\|_{\text{Rad}(X)}$, and hence, for each fixed n , the sequence $x_n^{(m)}$ converges to x_n as $m \rightarrow \infty$. Since $x_n^{(m)} \in Y_n$ for every n and all Y_n are closed, we have $x_n \in Y_n$ for all n , or equivalently $x \in Y$. \square

Proposition 2. *For every $1 \leq p \leq \infty$ and every Banach space E the space $\mathcal{H}_p^{\text{rad}}(E)$ endowed with the norm defined in (13) is a Banach space.*

Proof. Note that our space $\mathcal{H}_p^{\text{rad}}(E)$ is actually a subspace of $\text{Rad}(\mathcal{H}_p(E))$:

$$\mathcal{H}_p^{\text{rad}}(E) = \{(D_n)_n \in \text{Rad}(\mathcal{H}_p(E)) : D_n = a_n n^{-s}, a_n \in E\}.$$

Observe that each $F_n = \{a_n n^{-s} : a_n \in E\} \subseteq \mathcal{H}_p(E)$ is isometric to E and hence closed. This by Lemma 1 completes the proof. \square

The following result is crucial for the modern theory of Dirichlet series: For each $1 \leq p \leq \infty$ there is a constant $C_p > 1$ such that for any Banach space E and any Dirichlet series $\sum a_n n^{-s} \in \mathcal{H}_p(E)$ we for every N have

$$\left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_p(E)} \leq C_p \log N \left\| \sum_{n=1}^{\infty} a_n n^{-s} \right\|_{\mathcal{H}_p(E)}. \quad (14)$$

For $E = \mathbb{C}$ and $p = \infty$ this is a quantification of (9) given in [2, Lemma 1.1] (see also [26, Theorem 6.2.2]), and for $E = \mathbb{C}$ and $p = 1$ it is [6, Theorem 3.2]. For $E = \mathbb{C}$ and $1 < p < \infty$ the situation is even better, since by [1], the system $(n^{-s})_{n \in \mathbb{N}}$ then forms a Schauder basis of $\mathcal{H}_p(\mathbb{C})$; hence in this situation the log-term even disappears. The vector-valued case needs an alternative approach – see [14] for a proof which again is very much in the spirit of the starting case $E = \mathbb{C}$ and $p = \infty$ (so of Bohr's original ideas). For our new spaces $\mathcal{H}_p^{\text{rad}}(E)$ the situation is much simpler.

Proposition 3. *If $1 \leq p \leq \infty$, E is a Banach space and $\sum a_n n^{-s} \in \mathcal{H}_p^{\text{rad}}(E)$, then for every N we have*

$$\left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(E)} \leq \left\| \sum_{n=1}^{\infty} a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(E)}.$$

Moreover, the sequence of partial sums converges to $\sum_{n=1}^{\infty} a_n n^{-s}$ in $\mathcal{H}_p^{\text{rad}}(E)$.

Proof. Let us fix $\sum a_n n^{-s} \in \mathcal{H}_p^{\text{rad}}(E)$ and $N \in \mathbb{N}$. We define $\lambda_n = 1$ for $1 \leq n \leq N$ and $\lambda_n = 0$ for $n > N$. We use now the Contraction Principle (see e.g. [18, Theorem 12.2]) to get that for $N < M$,

$$\begin{aligned} \left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(E)} &= \int_0^1 \left\| \sum_{n=1}^N r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_p(E)} dt = \int_0^1 \left\| \sum_{n=1}^M \lambda_n r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_p(E)} dt \\ &\leq \int_0^1 \left\| \sum_{n=1}^M r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_p(E)} dt = \left\| \sum_{n=1}^M a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(E)}. \end{aligned}$$

By [18, Theorem 12.3] the series $\sum r_n a_n n^{-s}$ converges in $L_1([0, 1]; \mathcal{H}_p(E))$, hence

$$\left\| \sum_{n=1}^M r_n a_n n^{-s} \right\|_{L_1} \longrightarrow \left\| \sum_{n=1}^{\infty} r_n a_n n^{-s} \right\|_{L_1} \quad \text{as } M \rightarrow \infty.$$

By the very definition of the norm in $\mathcal{H}_p^{\text{rad}}(E)$ this gives the conclusion. \square

Our next result shows that for $1 \leq p < \infty$ the study of $\mathcal{H}_p^{\text{rad}}(\mathbb{C})$ reduces to the study of \mathcal{H}_2 (see also Proposition 5 for a vector-valued extension).

Proposition 4. *For $1 \leq p < \infty$ we have $\mathcal{H}_p^{\text{rad}}(\mathbb{C}) = \mathcal{H}_2$.*

Proof. For fixed $N \in \mathbb{N}$ use the definition of $\mathcal{H}_p^{\text{rad}}(\mathbb{C})$, Kahane's inequality (see e.g. [18, Theorem 11.1]), the definition of $\mathcal{H}_p(\mathbb{C})$, and finally Khinchin's inequality (see e.g. [18, Theorem 1.10]) in order to get

$$\begin{aligned} \left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(\mathbb{C})} &= \int_0^1 \left\| \sum_{n=1}^N r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_p(\mathbb{C})} dt \sim \left(\int_0^1 \left\| \sum_{n=1}^N r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_p(\mathbb{C})}^p dt \right)^{\frac{1}{p}} \\ &= \left(\int_0^1 \int_{\mathbb{T}^{\mathbb{N}}} \left| \sum_{\alpha} r_{p^{\alpha}}(t) a_{p^{\alpha}} z^{\alpha} \right|^p dz dt \right)^{\frac{1}{p}} = \left(\int_{\mathbb{T}^{\mathbb{N}}} \int_0^1 \left| \sum_{\alpha} r_{p^{\alpha}}(t) a_{p^{\alpha}} z^{\alpha} \right|^p dt dz \right)^{\frac{1}{p}} \\ &\sim \left(\int_{\mathbb{T}^{\mathbb{N}}} \left(\sum_{\alpha} |a_{p^{\alpha}} z^{\alpha}|^2 \right)^{\frac{p}{2}} dz \right)^{\frac{1}{p}} = \left(\int_{\mathbb{T}^{\mathbb{N}}} \left(\sum_{\alpha} |a_{p^{\alpha}}|^2 \right)^{\frac{p}{2}} dz \right)^{\frac{1}{p}} \\ &= \left(\sum_{\alpha} |a_{p^{\alpha}}|^2 \right)^{\frac{1}{2}} = \left(\sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

This gives the conclusion. \square

It is not surprising that in the vector-valued situation such a description of $\mathcal{H}_p^{\text{rad}}(E)$ is more involved. However, if the space E is nice enough we do have something. Let us recall [23, page 46] that a Banach lattice E is q -concave (with $1 \leq q < \infty$) if there exists a constant $C > 0$ such that for every choice $x_1, \dots, x_N \in E$

$$\left(\sum_{n=1}^N \|x_n\|^q \right)^{\frac{1}{q}} \leq C \left\| \left(\sum_{n=1}^N |x_n|^q \right)^{\frac{1}{q}} \right\|.$$

For a Banach lattice E we define $\widetilde{E(\ell_2)}$ to be the space of sequences $(x_n)_{n=1}^\infty$ in E such that

$$\|(x_n)_n\|_{\widetilde{E(\ell_2)}} = \sup_N \left\| \left(\sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}} \right\|_E < \infty.$$

The closure in $\widetilde{E(\ell_2)}$ of the subspace of finite sequences is denoted by $E(\ell_2)$. We remark that these two spaces coincide if and only if E is weakly sequentially complete (see [23, p. 46] for details).

Proposition 5. *If E is a Banach lattice that is q concave for some q , then $\mathcal{H}_p^{\text{rad}}(E) = E(\ell_2)$ for every $1 \leq p < \infty$.*

Proof. Let us first consider $a_1, \dots, a_N \in E$. By the very definition of the norms in $\mathcal{H}_p^{\text{rad}}(E)$ and $\mathcal{H}_p(E)$ and Kahane's inequality (that we apply twice) we have (with constants independent of N)

$$\begin{aligned} \left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(E)} &= \int_0^1 \left\| \sum_{n=1}^N r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_p(E)} dt \sim \left(\int_0^1 \left\| \sum_{n=1}^N r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_p(E)}^p dt \right)^{\frac{1}{p}} \\ &= \left(\int_0^1 \int_{\mathbb{T}^N} \left\| \sum_{\alpha} r_{p^\alpha}(t) a_{p^\alpha} z^\alpha \right\|_E^p dz dt \right)^{\frac{1}{p}} = \left(\int_{\mathbb{T}^N} \int_0^1 \left\| \sum_{\alpha} r_{p^\alpha}(t) a_{p^\alpha} z^\alpha \right\|_E^p dt dz \right)^{\frac{1}{p}} \\ &\sim \left(\int_{\mathbb{T}^N} \left(\int_0^1 \left\| \sum_{\alpha} r_{p^\alpha}(t) a_{p^\alpha} z^\alpha \right\|_E^p dt \right)^p dz \right)^{\frac{1}{p}}. \end{aligned}$$

But now, since E is q concave for some q , for each fixed $z \in \mathbb{T}^N$ we have by [23, Theorem 1.d.6]

$$\int_0^1 \left\| \sum_{\alpha} r_{p^\alpha}(t) a_{p^\alpha} z^\alpha \right\|_E^p dt \sim \left\| \left(\sum_{\alpha} |a_{p^\alpha} z^\alpha|^2 \right)^{\frac{1}{2}} \right\|_E^p = \left\| \left(\sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}} \right\|_E^p.$$

This, together with Proposition 3, yields the conclusion. \square

3 A reformulation of $S_p^{\text{rad}}(E)$

Maurizi and Queffélec showed in [24, Theorem 2.4] how S can be characterized in terms of bounds of the norm of the partial sums. A modification of their argument using [14] gives the following vector-valued version: For every $1 \leq p \leq \infty$ and every Banach space E

$$S_p(E) = \inf \left\{ \sigma > 0 \mid \exists c_\sigma \forall D = \sum_{n=1}^N a_n n^{-s} \in \mathcal{H}_p(E) : \sum_{n=1}^N \|a_n\| \leq c_\sigma N^\sigma \|D\|_{\mathcal{H}_p(E)} \right\}. \quad (15)$$

The original proof of [24, Theorem 2.4] for $E = \mathbb{C}$ uses two key tools. The proof of one inequality is based on a closed-graph argument using the fact that \mathcal{H}_p is Banach, and the proof of the converse inequality relies on (14). The results from the preceding section prepare us well to establish the following analogue of (15) within our setting.

Proposition 6. *For every $1 \leq p \leq \infty$ and Banach space E we have*

$$S_p^{\text{rad}}(E) = \inf \left\{ \sigma > 0 \mid \exists c_\sigma \forall D = \sum_{n=1}^N a_n n^{-s} \in \mathcal{H}_p^{\text{rad}}(E) : \sum_{n=1}^N \|a_n\| \leq c_\sigma N^\sigma \|D\|_{\mathcal{H}_p^{\text{rad}}(E)} \right\}. \quad (16)$$

Proof. To show one inequality, let us take $\sigma > S_p^{\text{rad}}(E)$. A closed-graph argument (here we need Proposition 2) gives that there exists $c_\sigma > 0$ such that

$$\sum_{n=1}^{\infty} \|a_n\| \frac{1}{n^\sigma} \leq c_\sigma \left\| \sum a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(E)}$$

for every $\sum a_n n^{-s} \in \mathcal{H}_p^{\text{rad}}(E)$. Then, for given $a_1, \dots, a_N \in E$ we have

$$\sum_{n=1}^N \|a_n\| \leq N^\sigma \sum_{n=1}^N \frac{\|a_n\|}{n^\sigma} \leq c_\sigma N^\sigma \left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(E)}.$$

Let us conversely fix now some $\sigma_0 > 0$ satisfying the inequality in Proposition 6, and choose $\sum a_n n^{-s} \in \mathcal{H}_p^{\text{rad}}(E)$. By Abel's summation and Proposition 3 we have, for any $\sigma > \sigma_0$,

$$\begin{aligned} \sum_{n=1}^N \|a_n\| \frac{1}{n^\sigma} &= \sum_{n=1}^N \|a_n\| \frac{1}{N^\sigma} + \sum_{n=1}^{N-1} \sum_{k=1}^n \|a_k\| \left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right) \\ &\leq c_{\sigma_0} N^{\sigma_0 - \sigma} \left\| \sum a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(E)} + \sum_{n=1}^{N-1} c_{\sigma_0} n^{\sigma_0} \left\| \sum a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(E)} \left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right). \end{aligned}$$

Standard computations following [2, Lemma 1.1] finally give

$$\sum_{n=1}^N \|a_n\| \frac{1}{n^\sigma} \leq c_{\sigma_0} \left\| \sum a_n n^{-s} \right\|_{\mathcal{H}_p^{\text{rad}}(E)} \left(1 + \sum_{n=1}^{\infty} \frac{\sigma}{n^{\sigma - \sigma_0 + 1}} \right).$$

Hence $S_p^{\text{rad}}(E) \leq \sigma$ and, since σ was arbitrary, the proof is completed. \square

As an immediate consequence we obtain the following

Corollary 7. *For every $1 \leq p \leq \infty$ and Banach space E we have $S_p^{\text{rad}}(E) \leq S_p(E)$.*

Proof. Let us take $\sigma > 0$ satisfying the condition in (15). Then for every choice of finitely many $a_1, \dots, a_N \in E$ and every $t \in [0, 1]$ we have

$$\sum_{n=1}^N \|a_n\| = \sum_{n=1}^N \|r_n(t) a_n\| \leq c_\sigma N^\sigma \left\| \sum_{n=1}^N r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_p(E)}.$$

Integration with respect to t and Proposition 6 give the conclusion. \square

4 Uniform a.s.-sign convergence versus absolut coverage for Hardy-type Dirichlet series

The following theorem is our first main result.

Theorem 8. *For every Banach space E and $1 \leq p \leq \infty$ we have*

$$S_p^{\text{rad}}(E) = 1 - \frac{1}{\cot E},$$

i.e., if a Dirichlet series $\sum a_n n^{-s} \in \mathfrak{D}(E)$ is a.e.-sign convergent in $\mathcal{H}_p(E)$, then $\sum_n \|a_n\|_E n^{-\sigma} < \infty$ for $\sigma > \sigma_0 := 1 - \frac{1}{\cot E}$, and σ_0 is best possible.

Note that this, in combination with (8), in particular shows that for each Banach space E and each $1 \leq p \leq \infty$ we have

$$S_p^{\text{rad}}(E) = S_p(E).$$

In view of Corollary 7 and (8), we only have to take care of the lower estimate.

In order to prove Theorem 8, we need the concept of m -homogeneous Dirichlet series (that was first suggested in [7] and whose set we denote by $\mathfrak{D}_m(E)$): Those $\sum a_n n^{-s}$ for which $a_n \neq 0$ only if n has exactly m prime divisors (counted with multiplicity, we denote this by $\Omega(n) = m$). Then we define

$$\mathcal{H}_{p,m}(E) \quad \text{and} \quad \mathcal{H}_{p,m}^{\text{rad}}(E),$$

to be the (closed) subspace of $\mathcal{H}_p(E)$ and $\mathcal{H}_p^{\text{rad}}(E)$, respectively, consisting of m -homogeneous Dirichlet series. It is well-known that for all $1 \leq p, q < \infty$ and m (see e.g. [10, Theorem 9.1] or [3])

$$\mathcal{H}_{p,m}(\mathbb{C}) = \mathcal{H}_{q,m}(\mathbb{C}). \quad (17)$$

We now can repeat the above program and define for every $m \in \mathbb{N}$, every $1 \leq p \leq \infty$ and every Banach space E

$$\begin{aligned} S_{p,m}(E) &:= \sup_{D \in \mathfrak{D}(E) \text{ } m\text{-hom.}} \sigma_a(D) - \sigma_{\mathcal{H}_p}(D) \\ S_{p,m}^{\text{rad}}(E) &:= \sup_{D \in \mathfrak{D}(E) \text{ } m\text{-hom.}} \sigma_a(D) - \sigma_{\mathcal{H}_p^{\text{rad}}(E)}^{\text{rad}}(D); \end{aligned}$$

obviously $S_{p,m}(E) \leq S_p(E)$ and $S_{p,m}^{\text{rad}}(E) \leq S_p^{\text{rad}}(E)$. Exactly as above (see the proof of Proposition 6), we may show that

$$S_{p,m}(E) = \inf \left\{ \sigma > 0 \mid \exists c_\sigma \forall D = \sum_{n=1}^N a_n n^{-s} \in \mathcal{H}_{p,m}(E) : \sum_{n=1}^N \|a_n\| \leq c_\sigma N^\sigma \|D\|_{\mathcal{H}_{p,m}(E)} \right\} \quad (18)$$

and

$$S_{p,m}^{\text{rad}}(E) = \inf \left\{ \sigma > 0 \mid \exists c_\sigma \forall D = \sum_{n=1}^N a_n n^{-s} \in \mathcal{H}_{p,m}^{\text{rad}}(E) : \sum_{n=1}^N \|a_n\| \leq c_\sigma N^\sigma \|D\|_{\mathcal{H}_{p,m}^{\text{rad}}(E)} \right\}. \quad (19)$$

Moreover, following the argument for Corollary 7 we have

$$S_{p,m}^{\text{rad}}(E) \leq S_{p,m}(E). \quad (20)$$

As a by product of our proof (see the end of Subsection 4.1) we are going to obtain the following result (for the analogue for finite dimensional spaces see Proposition 10):

Proposition 9. *For every infinite dimensional Banach space E and every m*

$$S_{p,m}^{\text{rad}}(E) = S_{p,m}(E) = 1 - \frac{1}{\cot E}.$$

We divide the proof of Theorem 8 into two separate cases: for finite and infinite dimensional spaces.

4.1 The finite dimensional case

For every finite dimensional Banach space E we have $\cot E = 2$. Then the following counterpart of (9) obviously implies the lower bound in Theorem 8.

Proposition 10. *For every finite dimensional Banach space E and every m*

$$S_{p,m}^{\text{rad}}(E) = S_{p,m}(E) = \begin{cases} \frac{1}{2} & \text{for } 1 \leq p < \infty \\ \frac{m-1}{2m} & \text{for } p = \infty. \end{cases}$$

For $p = \infty$ and $E = \mathbb{C}$ this result is due to Bohnenblust-Hille [7] and Hartman [19].

Proof. Since $S_{p,m}^{\text{rad}}(E)$ is invariant under renorming of E , we may assume that $E = \ell_2^k$ (i.e. \mathbb{C}^k with the euclidean norm). By (20) we need to show the proper lower bound for $S_{p,m}^{\text{rad}}(\ell_2^k)$ and the proper upper bound for $S_{p,m}(\ell_2^k)$. We start with the upper bound for $S_{p,m}(\ell_2^k)$: Assume first that $1 \leq p < \infty$. Given $a_1, \dots, a_N \in \ell_2^k$, we then conclude from the Cauchy-Schwarz inequality and Kahane's inequality that

$$\begin{aligned} \sum_{k=1}^N \|a_n\|_{\ell_2^k} &\leq N^{1/2} \left(\sum_{k=1}^N \|a_n\|_{\ell_2^k}^2 \right)^{1/2} = N^{1/2} \left(\int_{\mathbb{T}^{\mathbb{N}}} \left\| \sum_{|\alpha|=m} a_{p^\alpha} z_\alpha \right\|_{\ell_2^k}^2 \right)^{1/2} \\ &\sim N^{1/2} \left(\int_{\mathbb{T}^{\mathbb{N}}} \left\| \sum_{|\alpha|=m} a_{p^\alpha} z_\alpha \right\|_{\ell_2^k}^p \right)^{1/p} = N^{1/2} \left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_{p,m}(\ell_2^k)}, \end{aligned}$$

which by (18) shows what we want. Now for $p = \infty$ we conclude from Hölder's inequality and the polynomial Bohnenblust-Hille inequality (in the form of [15, Theorem 5.3]) that

$$\begin{aligned} \sum_{k=1}^N \|a_n\|_{\ell_2^k} &\leq N^{\frac{2m}{m-1}} \left(\sum_{k=1}^N \|a_n\|_{\ell_2^k}^{\frac{2m}{m-1}} \right)^{\frac{m+1}{2m}} \\ &\leq C^m N^{\frac{2m}{m-1}} \sup_{z \in \mathbb{T}^{\mathbb{N}}} \left\| \sum_{|\alpha|=m} a_{p^\alpha} z_\alpha \right\|_{\ell_2^k} = C^m N^{\frac{2m}{m-1}} \left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}_{\infty,m}(\ell_2^k)}, \end{aligned}$$

and again (18) gives the conclusion.

Let us turn to the lower bound of $S_{p,m}^{\text{rad}}(\ell_2^k)$: A simple argument shows that $S_{p,m}^{\text{rad}}(\mathbb{C}) \leq S_{p,m}^{\text{rad}}(\ell_2^k)$, so it remains to estimate $S_{p,m}^{\text{rad}}(\mathbb{C})$ from below. We again start with the case $1 \leq p < \infty$. Then we know from (17) that $S_{p,m}^{\text{rad}}(\mathbb{C}) = S_{2,m}^{\text{rad}}(\mathbb{C})$, and hence we may concentrate on the case $p = 2$.

Clearly $S_{2,m}^{\text{rad}}(\mathbb{C}) \geq S_{2,1}^{\text{rad}}(\mathbb{C})$, then we can assume that $\sigma > 0$ and $c_\sigma > 0$ are as in (18) with $p = 2$, $m = 1$ and $E = \mathbb{C}$. Hence by the prime number theorem there are constants $C_1, C_2 > 0$ such that

$$\frac{N}{\log N} \leq C_1 \sum_{\substack{n=1 \\ \Omega(n)=1}}^N 1 \leq c_\sigma N^\sigma \left\| \sum_{\substack{n=1 \\ \Omega(n)=1}}^N n^{-s} \right\|_{\mathcal{H}_{2,m}^{\text{rad}}(\mathbb{C})} \leq C_2 c_\sigma N^\sigma \left(\frac{N}{\log N} \right)^{1/2}, \quad (21)$$

and this is exactly what we need.

Finally, we consider the case $p = \infty$: We fix $\sigma > S_{\infty,m}^{\text{rad}}(\mathbb{C})$; by a standard closed graph argument there is a constant $c_\sigma > 0$ such that

$$\sum_{n=1}^{\infty} |a_n| \frac{1}{n^\sigma} \leq c_\sigma \left\| \sum_n a_n n^{-s} \right\|_{\mathcal{H}_{\infty}^{\text{rad}}(\mathbb{C})}. \quad (22)$$

We consider ε_α independent Rademacher random variables (i.e. each one taking values ± 1 with probability $1/2$) for $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = m$. By the Kahane-Salem-Zygmund inequality, as presented in [26, Theorem 5.3.4] there is a constant $C > 0$ such that

$$\int \sup_{z \in \mathbb{D}^N} \left| \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha|=m}} \varepsilon_\alpha(\omega) z^\alpha \right| d\omega \leq C \left(\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha|=m}} 1 \right)^{\frac{1}{2}} \sqrt{N \log m} \leq C N^{\frac{m+1}{2}} \sqrt{\log m}.$$

We consider now the polynomial $\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha|=m}} z^\alpha$ and denote by D_N the Dirichlet series associated to it by (6). Then

$$\|D_N\|_{\mathcal{H}_{\infty}^{\text{rad}}(\mathbb{C})} = \int_0^1 \sup_{z \in \mathbb{D}^N} \left\| \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha|=m}} r_{p^\alpha}(t) z^\alpha \right\| dt \leq C N^{\frac{m+1}{2}} \sqrt{\log m}.$$

With this and (22) we get that, for every N

$$\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha|=m}} \frac{1}{p^{\alpha\sigma}} \leq c_\sigma C N^{\frac{m+1}{2}} \sqrt{\log m}.$$

All we need now is a lower bound of $\sum_{|\alpha|=m} \frac{1}{p^{\alpha\sigma}}$. By a weak consequence of the Prime Number Theorem $p_j \sim j \log j$. Then for a fixed $\varepsilon > 0$ there is a constant $B > 0$ such that for all j we have $p_j \leq B j^{1+\varepsilon}$, hence

$$\sum_{|\alpha|=m} \frac{1}{p^{\alpha\sigma}} = \sum_{1 \leq j_1 \leq \dots \leq j_m \leq N} \frac{1}{(p_{j_1} \dots p_{j_m})^\sigma} \geq \frac{1}{B^m} \sum_{1 \leq j_1 \leq \dots \leq j_m \leq N} \frac{1}{(j_1 \dots j_m)^{(1+\varepsilon)\sigma}}.$$

Let us now observe that

$$\sum_{j_1, \dots, j_m=1}^N \frac{1}{(j_1 \dots j_m)^{(1+\varepsilon)\sigma}} \leq \sum_{1 \leq j_1 \leq \dots \leq j_m \leq N} m! \frac{1}{(j_1 \dots j_m)^{(1+\varepsilon)\sigma}}.$$

Then

$$\begin{aligned} \sum_{1 \leq j_1 \leq \dots \leq j_m \leq N} \frac{1}{(j_1 \dots j_m)^{(1+\varepsilon)\sigma}} &\geq \frac{1}{m!} \sum_{j_1, \dots, j_m=1}^N \frac{1}{(j_1 \dots j_m)^{(1+\varepsilon)\sigma}} \\ &= \frac{1}{m!} \left(\sum_{j=1}^N \frac{1}{j^{(1+\varepsilon)\sigma}} \right)^m \geq D \frac{N^m}{N^{(1+\varepsilon)\sigma m}}. \end{aligned}$$

This altogether gives that there is a constant K depending only on m such that

$$N^{m(1-(1+\varepsilon)\sigma)} \leq K \sqrt{\log m} N^{\frac{m+1}{2}},$$

which yields $\frac{m-1}{2m} \leq \sigma$ and gives the result. \square

4.2 The infinite dimensional case

Let us now prove Theorem 8 for infinite dimensional Banach spaces E . Once again, by Corollary 7 and equation (8), it suffices to check the following: Given an infinite dimensional Banach space E and $1 \leq p \leq \infty$ the following holds

$$1 - \frac{1}{\cot E} \leq S_p^{\text{rad}}(E). \quad (23)$$

Proof. For each fixed $t \in [0, 1]$ we have

$$\left\| \sum r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_p(E)} \leq \left\| \sum r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_\infty(E)}.$$

Integrating with respect to t we get that $\mathcal{H}_\infty^{\text{rad}}(E) \subset \mathcal{H}_p^{\text{rad}}(E)$ for every $1 \leq p \leq \infty$. Hence to find a lower bound for $S_p^{\text{rad}}(E)$ it is enough to get some lower estimate for $S_\infty^{\text{rad}}(E)$. What we are going to do is to work only with 1-homogeneous Dirichlet series, finding lower bounds for $S_{\infty,1}^{\text{rad}}(E)$. Recall from (18) that

$$S_{\infty,1}^{\text{rad}}(E) = \inf \left\{ \sigma > 0 : \exists c_\sigma \forall a_{p_1}, \dots, a_{p_N} \in E : \sum_{k=1}^N \|a_{p_k}\| \leq c_\sigma p_N^\sigma \left\| \sum_{k=1}^N a_{p_k} p_k^{-s} \right\|_{\mathcal{H}_\infty^{\text{rad}}(E)} \right\}.$$

On the other hand for each t ,

$$\left\| \sum_{k=1}^N r_{p_k}(t) a_{p_k} p_k^{-s} \right\|_{\mathcal{H}_\infty(E)} = \sup_{u \in \mathbb{T}^N} \left\| \sum_{k=1}^N r_{p_k}(t) a_{p_k} u_k \right\|_E = \sup_{w \in \mathbb{T}^N} \left\| \sum_{k=1}^N a_{p_k} w_k \right\|_E = \left\| \sum_{k=1}^N a_{p_k} p_k^{-s} \right\|_{\mathcal{H}_\infty(E)}.$$

Now, integrating on t we obtain

$$\left\| \sum_{k=1}^N a_{p_k} p_k^{-s} \right\|_{\mathcal{H}_\infty^{\text{rad}}(E)} = \int_0^1 \left\| \sum_{k=1}^N r_{p_k}(t) a_{p_k} p_k^{-s} \right\|_{\mathcal{H}_\infty(E)} dt = \left\| \sum_{k=1}^N a_{p_k} p_k^{-s} \right\|_{\mathcal{H}_\infty(E)}.$$

This means that $S_{\infty,1}^{\text{rad}}(E) = S_{\infty,1}(E)$. But from [13, p.554] we know that $S_{\infty,1}(E) = 1 - \frac{1}{\cot E}$ which completes the proof. \square

A brief analysis of the preceding proof shows that we also get Proposition 9 as a by-product.

5 Sharp estimates

By definition the x th Sidon constant for Dirichlet series is given by

$$S_\infty(x) := \sup_{(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}} \frac{\sum_{n \leq x} |a_n|}{\left\| \sum_{n \leq x} a_n n^{-s} \right\|_{\mathcal{H}_\infty(\mathbb{C})}}, \quad (24)$$

and its (almost) precise asymptotic is expressed in the following formula:

$$S_\infty(x) = \frac{\sqrt{x}}{e^{\left(\frac{1}{\sqrt{2}} + o(1)\right) \sqrt{\log x \log \log x}}}; \quad (25)$$

this results with weaker constants instead of $\frac{1}{\sqrt{2}}$ was proved in [22, Theorem 4.3], the lower estimate was given in [11, Théorème 1.1], and finally the upper estimate followed from the hypercontractivity of the Bohnenblust-Hille inequality in [12, Theorem 1]. In view of the characterization (15), equation (25) represents a sharp estimate of the largest possible width on which a Dirichlet series $D = \sum a_n n^{-s}$ converges uniformly but not absolutely. Given $x \geq 2$ and $1 \leq p \leq \infty$, asymptotically correct estimates for

$$S_p(x) := \sup_{(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}} \frac{\sum_{n \leq x} |a_n|}{\left\| \sum_{n \leq x} a_n \frac{1}{n^s} \right\|_{\mathcal{H}_p(\mathbb{C})}} \quad (26)$$

like (25) are unfortunately so far unknown for $p \neq 2$. For $p = 2$ we have $S_2(x) = \sqrt{x}$ by (7). An analogue of this definition in our probabilistic setting à la Hartman is (again $x \in \mathbb{N}$ and $1 \leq p \leq \infty$)

$$S_p^{\text{rad}}(x) := \sup_{(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}} \frac{\sum_{n \leq x} |a_n|}{\left\| \sum_{n \leq x} a_n \frac{1}{n^s} \right\|_{\mathcal{H}_p^{\text{rad}}(\mathbb{C})}}.$$

Proposition 6 and Theorem 8 (for $E = \mathbb{C}$) suggest the following analogue of (25). It can be seen as the definite result of Hartman's original question.

Theorem 11. *We have, as x tends to ∞*

$$S_p^{\text{rad}}(x) = \begin{cases} O(\sqrt{x}) & \text{for } 1 \leq p < \infty \\ \frac{\sqrt{x}}{e^{\left(\frac{1}{\sqrt{2}} + o(1)\right) \sqrt{\log x \log \log x}}} & \text{for } p = \infty. \end{cases}$$

The formula for $1 \leq p < \infty$ is an immediate consequence of Proposition 4. Let us deal with the case $p = \infty$. We first prove that for every x and p

$$S_p^{\text{rad}}(x) \leq S_p(x); \quad (27)$$

then the upper estimate for $S_\infty^{\text{rad}}(x)$ obviously follows from (25). By definition $S_p^{\text{rad}}(x)$ is the best constant $C > 0$ such that for all sequences $(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}$ we have $\sum_{n \leq x} |a_n| \leq C \left\| \sum_{n \leq x} a_n n^{-s} \right\|_{\mathcal{H}_p(\mathbb{C})}$. But for each $t \in [0, 1]$

$$\sum_{n \leq x} |a_n| = \sum_{n \leq x} |a_n r_n(t)| \leq C \left\| \sum_{n \leq x} r_n(t) a_n n^{-s} \right\|_{\mathcal{H}_p(\mathbb{C})},$$

so that (27) follows by integration. It remains to prove the lower estimate for $S_\infty^{\text{rad}}(x)$ in Theorem 11, and the arguments we give follow from an analysis of the proof for (25). Our presentation is close to that of [11] and also [26, Theorem 5.4.3], and it is mainly given for the sake of completeness. Before we start we need some preparation from analytic number theory.

Given $k \in \mathbb{N}$, define $\mathcal{J}(k) = \{\mathbf{j} = (j_1, \dots, j_k) \in \mathbb{N}_0^k : 1 \leq j_1 \leq \dots \leq j_k\}$, and for any sequence $z = (z_n)$ of complex numbers and any $\mathbf{j} \in \mathcal{J}(k)$ let $z_{\mathbf{j}} = z_{j_1} \cdots z_{j_k}$. Moreover, for $x > 2$ and $2 < y \leq x$, choose $\ell \in \mathbb{N}$ such that $p_\ell \leq y < p_{\ell+1}$ (note that with the usual notation from number theory $\ell = \pi(y)$). With x and ℓ define the index set

$$J^-(x; y) = \{\mathbf{j} = (j_1, \dots, j_k) \in \mathcal{J}(k) : k \in \mathbb{N}, p_{\mathbf{j}} \leq x, j_k \leq \ell\}.$$

Note first that $2^{\text{length}(\mathbf{j})} \leq p_{\mathbf{j}} \leq x$ for every $\mathbf{j} \in J^-(x; y)$, hence the maximal length

$$L := \max \{\text{length}(\mathbf{j}) : \mathbf{j} \in J^-(x; y)\} \leq \frac{\log x}{\log 2}. \quad (28)$$

The asymptotic behavior of the function $|J^-(x; y)|$ is very well described by the so called Dickmann function $\varrho : [0, \infty[\rightarrow \mathbb{R}$ which is uniquely determined through the following conditions:

- ϱ is differentiable on $]1, \infty[$ where it satisfies the differential equation

$$u\varrho'(u) + \varrho(u-1) = 0.$$

- $\varrho(u) = 1$ for all $0 \leq u \leq 1$, and ϱ is continuous at 1.

For this definition see e.g. [27, III.5, p. 365, 370]. We need the following two asymptotic estimates; the first one can be found in [27, III.5.5, Corollary 9.3] (see also [21, Eq. (1.8)]), and the second in [21, Eq. (1.7)]:

- Given $\varepsilon > 0$, there is $C = C(\varepsilon) > 0$ such for all x, y with $x > 2$ and $e^{(\log \log x)^{\frac{5}{3}+\varepsilon}} \leq y \leq x$

$$\frac{1}{C}x\varrho(u) \leq |J^-(x; y)| \leq Cx\varrho(u), \quad (29)$$

where here (and in the sequel) $u = \frac{\log x}{\log y}$.

- For $u \rightarrow \infty$:

$$\log \varrho(u) = -u \log u (1 + o(1)). \quad (30)$$

We are now ready to start the

Proof of the lower estimate of $S^{\text{rad}}(x)$ in Theorem 11. Fix $x > 2$, and choose some $2 < y \leq x$ together with some $\ell \in \mathbb{N}$ for which $p_\ell \leq y < p_{\ell+1}$ (later it will turn out that the optimal choice for y in fact is $y = e^{\frac{1}{\sqrt{2}}\sqrt{\log x \log \log x}}$). The general strategy will be to apply in a first step the Kahane-Salem-Zygmund inequality [26, Theorem 5.3.4] in order to get

$$\sqrt{\frac{|J^-(x; y)|}{y \log \log x}} \leq K S^{\text{rad}}(x) \quad (31)$$

for some universal K and then in a second step to optimize y with analytic number theory.

Define the finite Dirichlet series

$$D_{x,y} = \sum_{\mathbf{j} \in J^-(x; y)} \frac{1}{p_{\mathbf{j}}^s},$$

which obviously has length $\leq x$. Clearly

$$\sum_{\mathbf{j} \in J^-(x; y)} 1 = |J^-(x; y)|,$$

and therefore our aim for the proof of (31) will be to show

$$\|D_{x,y}\|_{\mathcal{H}_{\infty}^{\text{rad}}} \leq K \sqrt{y |J^-(x;y)| \log \log x}. \quad (32)$$

By Bohr's fundamental lemma (see e.g. [26, Theorem 4.4.2]) we have

$$\|D_{x,y}\|_{\mathcal{H}_{\infty}^{\text{rad}}} = \int_0^1 \sup_{z \in \mathbb{T}^\ell} \left| \sum_{\mathbf{j} \in J^-(x;y)} r_{p\mathbf{j}}(t) z_{\mathbf{j}} \right| dt.$$

Hence by (28) we deduce from the Kahane-Salem-Zygmund inequality (see the version given in [26, Theorem 5.3.4]) that

$$\|D_{x,y}\|_{\mathcal{H}_{\infty}^{\text{rad}}} \leq K \sqrt{\ell |J^-(x;y)| \log \log x}.$$

But trivially $\ell \leq y$ which gives (32) and hence (31). To finish the proof the number theoretical results from (29) and (30) enter the game. Assume that $y = e^{\alpha \sqrt{\log x \log \log x}}$, where $\alpha > 0$ will be specified later (as already noted it will turn out that the perfect choice is $\alpha = \frac{1}{\sqrt{2}}$). Put

$$u := \frac{\log x}{\log y} = \frac{1}{\alpha} \frac{\log x}{\log \log x}.$$

A simple calculation then gives

$$u \log u = \frac{1}{2\alpha} \sqrt{\log x \log \log x} (1 + o(1)). \quad (33)$$

Note also that, taking for example $\varepsilon = 1$, y lies in the interval of validity of inequality (29). Then we have:

$$\begin{aligned} S^{\text{rad}}(x) &\stackrel{(31),(29)}{\geq} K_1 \sqrt{\frac{x}{\log \log x}} \sqrt{\frac{\varrho(u)}{y}} \\ &\stackrel{\text{def. of } y}{=} K_1 \sqrt{\frac{x}{\log \log x}} e^{\frac{\log \varrho(u)}{2}} e^{-\frac{\alpha}{2} \sqrt{\log x \log \log x}} \\ &\stackrel{(30),(33)}{\geq} K_2 \sqrt{\frac{x}{\log \log x}} e^{-\left(\frac{1}{4\alpha} + \frac{\alpha}{2} + o(1)\right) \sqrt{\log x \log \log x}} = K_2 \sqrt{x} e^{-\left(\frac{1}{4\alpha} + \frac{\alpha}{2} + o(1)\right) \sqrt{\log x \log \log x}}. \end{aligned}$$

Minimizing $\frac{1}{4\alpha} + \frac{\alpha}{2}$ yields the optimal parameter $\alpha = \frac{1}{\sqrt{2}}$, and we finally arrive at the desired lower estimate for $S^{\text{rad}}(x)$ in Theorem 11. \square

Again it is possible to graduate the result from Theorem 11 along m -homogeneous polynomials. As in (24) and (26) we may define

$$S_{p,m}(x) \quad \text{and} \quad S_{p,m}^{\text{rad}}(x), \quad x \in \mathbb{N}$$

replacing \mathcal{H}_p by $\mathcal{H}_{p,m}$ as well as $\mathcal{H}_p^{\text{rad}}$ by $\mathcal{H}_{p,m}^{\text{rad}}$, and again we see that $S_{p,m}^{\text{rad}}(x) \leq S_{p,m}(x)$. A careful analysis of [2, Theorem 1.4] and [24, Theorem 3.1] (see also [17]) proves

$$S_{\infty,m}(x) = O\left(\frac{x^{\frac{m-1}{2m}}}{(\log x)^{m-1}}\right), \quad (34)$$

and then the following m -homogeneous variant of Theorem 11 comes naturally.

Theorem 12. *We have, as x tends to ∞*

$$S_{p,m}^{\text{rad}}(x) = \begin{cases} O(\sqrt{x}) & \text{for } 1 \leq p < \infty \\ O\left(\frac{x^{\frac{m-1}{2m}}}{(\log x)^{m-1}}\right) & \text{for } p = \infty. \end{cases}$$

Only the lower estimates have to be checked. For the case $1 \leq p < \infty$ argue as in the proof of Theorem 10. For $p = \infty$ analyse again the proof of the lower estimate in (34).

6 Appendix: On the abscissa of a.s.-sign convergence

One of the remarkable results of the work of Hartman [19] was that, unlike the classical strips ((2) and (3)), the maximal width of the two strips of the a.s.-sign convergence coincide ((4) and (5)). We already pointed out (10) that this result fits in our point of view and in fact follows from our Theorem 8.

We wonder now what happens with the abscissas of a.s.-sign convergence and absolute convergence for vector-valued Dirichlet series. Will it again be the case that the maximal distance between these two is the same as the maximal width for the abscissa of a.s.-sign uniform and absolute convergence? We answer this question now. Let us introduce some notation just for this appendix; for a given Banach space E we consider the numbers

$$S_{c \rightarrow a}^{\text{rad}}(E) := \sup_{D \in \mathfrak{D}(E)} \sigma_a(D) - \sigma_c^{\text{rad}}(D)$$

$$S_{u \rightarrow a}^{\text{rad}}(E) := \sup_{D \in \mathfrak{D}(E)} \sigma_a(D) - \sigma_u^{\text{rad}}(D)$$

By $S_{m,c \rightarrow a}^{\text{rad}}(E)$ and $S_{m,u \rightarrow a}^{\text{rad}}(E)$ we denote their graduations along the homogeneity $m \in \mathbb{N}$, defined in the obvious way. Observe that $S_{u \rightarrow a}^{\text{rad}}(E)$ and $S_{m,u \rightarrow a}^{\text{rad}}(E)$ are just the $S_{\infty}^{\text{rad}}(E)$ and $S_{\infty,m}^{\text{rad}}(E)$ that we considered before.

Obviously we have the trivial estimates

$$S_{m,u \rightarrow a}^{\text{rad}}(X) \leq S_{u \rightarrow a}^{\text{rad}}(X) \quad \text{and} \quad S_{m,c \rightarrow a}^{\text{rad}}(X) \leq S_{c \rightarrow a}^{\text{rad}}(X) \quad (35)$$

as well as

$$S_{u \rightarrow a}^{\text{rad}}(X) \leq S_{c \rightarrow a}^{\text{rad}}(X) \quad \text{and} \quad S_{m,u \rightarrow a}^{\text{rad}}(X) \leq S_{m,c \rightarrow a}^{\text{rad}}(X). \quad (36)$$

Our aim is to show that for every Banach space E we have

$$S_{u \rightarrow a}^{\text{rad}}(E) = S_{c \rightarrow a}^{\text{rad}}(E) = S_{m,c \rightarrow a}^{\text{rad}}(E) = 1 - \frac{1}{\cot(E)}; \quad (37)$$

and if E is infinite-dimensional, then we can also put $S_{m,u \rightarrow a}^{\text{rad}}(E)$ within the previous inequalities. The equalities for $S_{u \rightarrow a}^{\text{rad}}(E)$ and $S_{m,u \rightarrow a}^{\text{rad}}(E)$ follow from Theorem 8 and Proposition 9 with $p = \infty$. We once again mention that the scalar case $E = \mathbb{C}$ is due to Bohr, Bohnenblust-Hille and Hartman. We consider again the space

$$\text{Rad}(E) := \left\{ a = (a_n) \in E^{\mathbb{N}} : \sum_{n=1}^{\infty} a_n r_n \in L_1([0, 1]; E) \right\}$$

which together with the norm

$$\|(a_n)_n\|_{\text{Rad}(E)} := \int_0^1 \left\| \sum_{n=1}^{\infty} a_n r_n(t) \right\|_E dt$$

forms a Banach space. Recall that $(a_n)_n$ belongs to $\text{Rad}(E)$ if and only if $\sum_{n=1}^{\infty} a_n \varepsilon_n$ converges for almost all choices of signs ε_n . In particular,

$$\sigma_c^{\text{rad}}(D) = \inf \left\{ \sigma \in \mathbb{R} : \left(\frac{a_n}{n^\sigma} \right) \in \text{Rad}(E) \right\}. \quad (38)$$

Let us note that the key ingredient to get descriptions of the width of the strip in the spirit of Maurizi-Queffélec (see (15), (16), (18) and (19)) is to have a norm that provides a proper control of the size of the partial sums, like in (14). Observe that now, by Kahane's contraction principle, we have that for each N

$$\|(a_n)_{n=1}^N\|_{\text{Rad}(E)} \leq \|(a_n)\|_{\text{Rad}(E)}.$$

Proceeding as in Proposition 6, using this instead of Proposition 3, we obtain

$$S_{c \rightarrow a}^{\text{rad}}(E) = \inf \left\{ \sigma > 0 \mid \exists c_\sigma \forall D = \sum_{n=1}^N a_n n^{-s} \in \mathfrak{D}(E) : \sum_{n=1}^N \|a_n\| \leq c_\sigma N^\sigma \|(a_n)_{n=1}^N\|_{\text{Rad}(E)} \right\} \quad (39)$$

$$S_{m,c \rightarrow a}^{\text{rad}}(E) = \inf \left\{ \sigma > 0 \mid \exists c_\sigma \forall D = \sum_{n=1}^N a_n n^{-s} \in \mathfrak{D}_m(E) : \sum_{n=1}^N \|a_n\| \leq c_\sigma N^\sigma \|(a_n)_{n=1}^N\|_{\text{Rad}(E)} \right\} \quad (40)$$

Note that in the scalar case $E = \mathbb{C}$, by Khinchine's inequality, we see that

$$S_{c \rightarrow a}^{\text{rad}}(\mathbb{C}) = \inf \left\{ \sigma > 0 \mid \exists c_\sigma \forall D = \sum_{n=1}^N a_n n^{-s} \in \mathfrak{D}(\mathbb{C}) : \sum_{n=1}^N |a_n| \leq c_\sigma N^\sigma \left(\sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}} \right\},$$

and hence, applying the Cauchy-Schwarz inequality, we obtain Hartman's result $S_{c \rightarrow a}^{\text{rad}}(\mathbb{C}) = \frac{1}{2}$.

Finally to complete the proof of (37) it only remains to show the following

Theorem 13. *For every Banach space E and every $m \in \mathbb{N}$*

$$S_{c \rightarrow a}^{\text{rad}}(E) = S_{m,c \rightarrow a}^{\text{rad}}(E) = 1 - \frac{1}{\cot(E)}.$$

Proof. We begin with the equality for $S_{c \rightarrow a}^{\text{rad}}(E)$. By (35) and the lower estimate for $S_{u \rightarrow a}^{\text{rad}}(E)$ from (37) we have to check

$$S_{c \rightarrow a}^{\text{rad}}(E) \leq 1 - \frac{1}{\cot(E)}. \quad (41)$$

Take $q > \cot(E)$. Then for each finite Dirichlet series $D = \sum_{n=1}^N a_n \frac{1}{n^s} \in \mathfrak{D}(E)$ by Hölder's inequality

$$\sum_{n=1}^N \|a_n\| \leq N^{\frac{1}{q'}} \left(\sum_{n=1}^N \|a_n\|^q \right)^{\frac{1}{q}} \leq C_q(E) N^{\frac{1}{q'}} \|(a_n)\|_{\text{Rad}(E)}.$$

Hence we obtain from (39) that $S_{c \rightarrow a}^{\text{rad}} \leq 1 - \frac{1}{q}$, the conclusion.

We finish by giving the argument for $S_{m,c \rightarrow a}^{\text{rad}}(E)$. If E is infinite-dimensional, by (35) and (36), the result for $S_{m,u \rightarrow a}^{\text{rad}}(E)$ from (37), and (41) we have

$$1 - \frac{1}{\cot(E)} = S_{m,u \rightarrow a}^{\text{rad}}(E) \leq S_{m,c \rightarrow a}^{\text{rad}}(E) \leq S_{c \rightarrow a}^{\text{rad}}(E) = 1 - \frac{1}{\cot(E)}.$$

On the other hand, if E is finite dimensional we can argue as in (21) to show that $S_{1,c \rightarrow a}^{\text{rad}}(\mathbb{C}) \geq 1/2$. This completes the proof. \square

In the scalar case $E = \mathbb{C}$ and in view of Proposition 38, it is again possible to graduate $S_{c \rightarrow a}^{\text{rad}}(\mathbb{C})$ and $S_{m,c \rightarrow a}^{\text{rad}}(\mathbb{C})$, respectively, along the length of the Dirichlet polynomials. As in (24), for $x \geq 1$ we define

$$S_{c \rightarrow a}^{\text{rad}}(x) := \sup_{(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}} \frac{\sum_{n \leq x} |a_n|}{\|(a_n)_{n \leq x}\|_{\text{Rad}(E)}},$$

and similarly $S_{m,c \rightarrow a}^{\text{rad}}(x)$. Then by Khinchine's inequality and the Cauchy-Schwarz inequality we obviously have

$$S_{c \rightarrow a}^{\text{rad}}(x) = S_{m,c \rightarrow a}^{\text{rad}}(x) \sim \sqrt{x}.$$

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